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TECHNICAL REPORT 703-5

ON FLOW PAST A SUPERCAVITATING CASCADE OF CAMBERED BLADES

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NOTATION

A,B	Constants
c	Chord length
C_L	Lift coefficient
C_D	Drag coefficient
d	Spacing between the adjacent blades
i	$= \sqrt{-1}$
k	Camber coefficient
ℓ	Cavity length
P_1	Pressure at far upstream
P_c	Cavity pressure
s	Chord length in the transformed plane
U	Uniform free stream velocity
U_c	Constant cavity speed
u,v	Perturbed velocity components in the x- and y-direction respectively
w	$=(u - iv)/U_c$ complex velocity function
x,y	Cartesian coordinates of physical plane
y_f	Profile of the wetted lower blade surface
y_c	Cavity shape
α_1	Angle of attack
α_2	Flow angle with respect to x-direction at far downstream
γ	Stagger angle

ξ, η	Coordinates in the transformed plane
ζ	$= \xi + i\eta$
σ	Cavitation number
$R\ell$, Im	Denote the real and imaginary part of the designated quantity

INTRODUCTION

In the study of turbomachines and propellers, one often approximates the actual conditions by considering a two-dimensional flow past an infinite lattice of indentical blades. The problem of a fully wetted cascade is well-known, but does not seem to be extensively treated for cascades with cavitation. Generally the occurrence of cavitation is promoted by high fluid velocities and relatively low ambient pressures. Both of these conditions prevail in inducer pumps for missiles for example. The purpose of this report is to provide an approximate method for calculating flow of incompressible fluid through a cascade of cambered blades with finite cavities.

The difficulty in this case can be appreciated by considering the much simpler problem of incompressible flow past a single supercavitating hydrofoil. The only exact analytical method known, based on certain artificial models of cavity termination, is the hodograph technique, which is rather complex. To solve such complex problems, one usually introduces suitable approximations. The most convenient of them is the linearized, closed-cavity theory of Tulin (1953). Realizing the complication involved, a similar approximation is made in the present analysis. The problem of finding the cascade flow characteristics is then reduced to one of quadratures.

It is almost needless to say that the method is inherently limited in its applications because it is based on the assumption of small thickness of the blades being treated and relatively small disturbances being generated. However, in practice, the pump, turbine or propeller blades are thin and consequently the linearized results may be useful as a guide in the design of turbomachines and propellers.

GENERAL FORMULATIONS

Consider the flow schematically illustrated in Figure 1. The cascade consists of an infinite array of identical cambered blades having a stagger angle γ and an angle of attack α_1. The mean chord length of each blade is c and the spacing of the blades in the direction of the stagger angle is d. The flow is turned by the cascade from its original horizontal direction and velocity U_1 at upstream infinity to the direction α_2 and velocity U_2 at a location far downstream.

In the analysis to be developed, it is assumed that the blade and cavity, with length $\ell > c$, are equivalent to a slender body which causes only small disturbances in an otherwise uniform stream. As a first approximation the boundary conditions may be applied on the x-axis instead of on the slender body. If the cavities are assumed to be detached from the sharp leading and trailing edges of each blade, the linearized boundary conditions on the wetted lower surface of each cavitating blade may be expressed as:

$$\frac{v}{U_c} \doteq -\alpha_1 + \frac{dy_f}{dx} \qquad 0 < x < c \quad , \quad y = 0^-$$

On the cavity the boundary condition is:

$$\frac{u}{U_c} = 0 \qquad 0 < x < \ell \quad , \quad y = 0^+ \quad \text{and} \quad c < x < \ell \quad , \quad y = 0^-$$

At upstream infinity

$$\frac{u}{U_c} = \frac{1}{\sqrt{1 + \sigma}} - 1 \approx -\frac{\sigma}{2} \quad , \quad \frac{v}{U_c} = 0$$

in which

u,v = Perturbed velocity components in the x- and y-axis respectively

U_c = Constant cavity speed = $\sqrt{1 + \sigma}\, U_1$

y_f = Profile of lower blade surface

σ = Cavitation number = $\dfrac{P_1 - P_c}{\frac{1}{2}\rho U_1^2} << 1$, a constant

ρ = fluid density

In order to insure that the cavity is closed it is also required that

$$\oint_{\text{body}} \frac{v}{U_c}\, dx = 0$$

The linearized problem may now be stated as follows: Given a cascade defined by γ, ℓ/d, c/d, α_1 and y_f it is required to find the harmonic function $w(x,y) = u/U_c - i\, v/U_c$ on an infinitely connected domain which satisfies the mixed boundary conditions on the body, the conditions at upstream infinity and the closure condition.

The problem as formulated may be greatly simplified, (since the flow is periodic) with the aid of the conformal transformation

$$z = x + iy = \frac{d}{2\pi}\left[e^{-i\gamma} \ln \frac{1 - \zeta/\zeta_1}{1 - \zeta/\zeta_2} + e^{i\gamma} \ln \frac{1 - \zeta/\bar{\zeta}_1}{1 - \zeta/\bar{\zeta}_2}\right] \qquad [1]$$

which maps the multiple-connected region in the z-plane onto the $\zeta(= \xi + i\eta)$ plane as shown in Figures 2 and 3. The function has branch points at $\zeta_1 = r_1 e^{i(\pi/2 - \varphi)}$ and $\zeta_2 = r_2 e^{i(\pi/2 + \varphi)}$ corresponding respectively to up- and down-stream infinity in the physical z-plane. The line between ζ_1 and ζ_2 is a branch cut of the mapping function. By crossing the cut, the value of the logarithmic function in Equation [1] changes by $2\pi i$ or $-2\pi i$. Each Riemann sheet in the ζ-plane corresponds to the flow region over a different blade-cavity body in the cascade. The leading edge of the blade-cavity body is mapped to the origin of the ζ-plane and the trailing edge to a point at infinity. The juncture of the lower blade surface and cavity maps to $\zeta = -s$.

The linearized cascade problem is now reduced to that of finding a harmonic function $w(\zeta)$ in the ζ-plane which satisfies the following boundary conditions:

$$\left.\begin{array}{l}
\text{On the wetted surface} \\[4pt]
-\mathrm{Im}\ w = -\alpha_1 + \dfrac{dy_f}{dx}(\xi) \qquad -s < \xi < 0, \quad \eta = 0 \\[8pt]
\text{On the cavity} \\[4pt]
R\ell\ w = 0 \qquad -\infty < \xi < -s \quad \text{and} \quad 0 < \xi < \infty \quad \eta = 0 \\[8pt]
\text{At upstream infinity, i.e., at } \zeta = \zeta_1 \\[4pt]
\mathrm{Im}\ w = 0, \quad R\ell\ w = -\dfrac{\sigma}{2}
\end{array}\right\} \qquad [2]$$

and the closure condition

$$\oint_{\text{body}} \frac{v}{U_c}(x)dx = -\mathrm{Im} \oint_{\text{body}} w(z)dz = -\mathrm{Im} \oint_{\text{body}} w(\zeta)\frac{dz}{d\zeta}d\zeta = 0 \qquad [3]$$

where $R\ell$ and Im denote the real and imaginary part of the designated quantity. This boundary-value problem resembles closely that of the flow past an isolated supercavitating hydrofoil and is a special case of the Riemann-Hilbert problem for a half-plane. (Muskhelishvili, 1953).

Before giving the details of the general solution it is expedient, first, to introduce the cascade parameters, given originally by Cohen and Sutherland (1958), which characterize the geometry of the cascade. These are:

$$\frac{\ell}{d} = \frac{1}{\pi}\left(\cos\gamma \,\ell n \frac{r_2}{r_1} + 2\varphi \sin\gamma\right) \qquad [4]$$

$$\frac{c}{d} = \frac{1}{\pi}\left\{\frac{\cos\gamma}{2} \,\ell n \frac{1 + 2(s/r_1)\sin\varphi + (s/r_1)^2}{1 - 2(s/r_2)\sin\varphi + (s/r_2)^2}\right.$$

$$\left. + \sin\gamma\left[\tan^{-1}\frac{-(s/r_1)\cos\varphi}{1 + (s/r_1)\sin\varphi} - \tan^{-1}\frac{(s/r_2)\cos\varphi}{1 - (s/r_2)\sin\varphi}\right]\right\} \qquad [5]$$

where

$$r_1 = \frac{2Q^{3/2}\sqrt{d}}{(Q + \cos\gamma)\sqrt{\cos\gamma}\cosh\theta}$$

$$r_2 = r_1\left(\frac{Q + \cos\gamma}{\sinh\theta}\right)^2$$

$$Q = (\cosh^2\theta - \sin^2\gamma)^{\frac{1}{2}}$$

$$\varphi = \tan^{-1}\left(\frac{\sin\gamma}{Q}\right)$$

$$\theta = \ell n\left(\frac{b}{a}\right), \text{ a and b are arbitrary constants}$$

Equation [4] gives the length of the slender (blade-cavity) body as functions of given parameters γ, d and b/a. Equation [5] determines the value of chord length s in the transformed ζ-plane for given values of γ, d, b/a and solidity c/d.

SOLUTION OF THE BOUNDARY-VALUE PROBLEM

The general solution of the reduced boundary-value problem stated in the previous section, may be shown to be of the form

$$w(\zeta) = -\frac{1}{\pi} H(\zeta) \int_{-\infty}^{\infty} \frac{iw(t)}{H(t)} \frac{dt}{t-\zeta} + H(\zeta)\, P(\zeta) \qquad [6]$$

where

$H(\zeta)$ is the fundamental solution which depends on the flow behavior at the edge points $\zeta = 0$ and $\zeta = -s$,

$P(\zeta)$ is a rational function which depends on the flow behavior at the edge points and $\pm\infty$.

The first term of Equation [6] is the particular solution which satisfies the mixed boundary conditions on the real ξ-axis while the second term is the general solution of the corresponding homogeneous problem.

In accordance with the linearized formulation it is required that $w(\zeta)$ satisfies the following conditions as in the case of flow past a single foil (Tulin, 1964):

$$\begin{aligned} &w(\zeta) \sim \zeta^{-\frac{1}{2}} && \text{at } \zeta = 0 \\ &w(\zeta) \text{ is finite} && \text{at } \zeta = -s \quad \text{(smooth juncture condition)} \quad [7] \\ &w(\zeta) \sim \zeta && \text{at } \zeta = \infty \end{aligned}$$

The functions $H(\zeta)$ and $P(\zeta)$ then take the forms

$$H(\zeta) = i\sqrt{\frac{\zeta + s}{\zeta}}$$
[8]
$$P(\zeta) = A\zeta + B$$

where A and B are real constants. The general solution of the boundary-value problem, in this case, becomes

$$w(\zeta) = i\sqrt{\frac{\zeta + s}{\zeta}}\left[-\frac{1}{\pi}\int_{-s}^{0}\frac{\left(-\alpha_1 + \frac{dy_f}{dx}(t)\right)}{t - \zeta}\sqrt{\frac{-t}{t + s}}\,dt + A\zeta + B\right] \qquad [9]$$

which should, in addition, satisfy

uniform flow condition at upstream infinity

$$R\ell\ w(\zeta_1) = -\frac{\sigma}{2} \qquad [10]$$

$$\mathrm{Im}\ w(\zeta_1) = 0 \qquad [11]$$

and the closure condition

$$-\mathrm{Im} \oint_{\text{body}} w(\zeta)\, \frac{dz}{d\zeta} = -\mathrm{Im} \oint_{\substack{\text{about} \\ \zeta_1 \text{ and } \zeta_2}} w(\zeta)\, \frac{dz}{d\zeta}\, d\zeta = 0, \text{ or}$$

$$\mathrm{Im}\left[2\pi i\, \frac{d}{2\pi}\, e^{-i\gamma}\left(w(\zeta_1) - w(\zeta_2)\right)\right] = -d\, R\ell\left[e^{-i\gamma}\left(\frac{\sigma}{2} + w(\zeta_2)\right)\right] = 0 \qquad [12]$$

Equations [10], [11] and [12] give uniquely, the values of A, B and σ. For a prescribed oncoming flow direction, body shape, cavity length and cascade geometry the flow field $w(\zeta)$ is, therefore, completely determined.

In the case of flat plate cascade, i.e., $dy_f/dx = 0$, the complex velocity field is of the form

$$w(\zeta) = i\sqrt{\frac{\zeta + s}{\zeta}}\left[\alpha\left(1 - \sqrt{\frac{\zeta}{\zeta + s}}\right) + A\zeta + B\right]$$

which, together with Equations [10], [11] and [12], may be shown to yield a solution identical to that given by Cohen and Sutherland (1958).

The lift, experienced by the blade as a result of fluid flow, is given to a first order of approximation, in non-dimensional form, by

$$C_L = \frac{\text{Lift}}{\frac{1}{2}\rho U_c^2 c} = -\frac{2}{c} R\ell \oint_{\text{body}} w(z)dz = -\frac{2}{c} R\ell \oint_{\text{body}} w(\zeta)\frac{dz}{d\zeta}d\zeta$$

$$= \frac{2}{c/d} \text{Im}\left[e^{-i\gamma}\left(\frac{\sigma}{2} + w(\zeta_2)\right)\right] = \frac{2}{(c/d)\cos\gamma} \text{Im } w(\zeta_2) \qquad [13]$$

Within the framework of linearized theory the drag coefficient may be expressed as

$$C_D = \frac{\text{Drag}}{\frac{1}{2}\rho U_c^2 c} = -\frac{1}{c}\text{Im}\int_0^c [w(z)]^2 dz = -\frac{1}{c}\text{Im}\int_0^{-s}[w(\zeta)]^2 \frac{dz}{d\zeta}d\zeta \qquad [14]$$

Similarly the upper and lower cavity shapes can be approximated by the following equations:

$$\left.\begin{aligned} y_{c_{\text{upper}}} &= \int_0^x \frac{dy_c}{dx} = -\text{Im}\int_0^x w(x,0^+)dx = -\text{Im}\int_0^{x(\xi,0)} w(\zeta)\frac{dz}{d\zeta}d\zeta \\ y_{c_{\text{lower}}} &= \int_c^x \frac{dy_c}{dx} = -\text{Im}\int_c^x w(x,0^-)dx = -\text{Im}\int_{-s}^{x(\xi,0)} w(\zeta)\frac{dz}{d\zeta}d\zeta \end{aligned}\right\} \qquad [15]$$

The exit flow velocity U_2/U_c and angle α_2 are approximately given by

$$\frac{U_2}{U_c} = \left\{\left[\left(1-\frac{\sigma}{2}\right) + R\ell\ w(\zeta_2)\right]^2 + \left[-\text{Im } w(\zeta_2)\right]^2\right\}^{\frac{1}{2}}$$

$$= \left\{\left[(1-\sigma) + (-\text{Im } w(\zeta_2)\tan\beta\right]^2 + \left[-\text{Im } w(\zeta_2)\right]^2\right\}^{\frac{1}{2}} \qquad [16]$$

and

$$\alpha_2 = \sin^{-1}\left[\frac{-\mathrm{Im}\ w(\zeta_2)}{U_2/U_c}\right] \doteq -\mathrm{Im}\ w(\zeta_2)$$

$$= -\frac{1}{2} C_L \left(\frac{c}{d}\right) \cos\gamma \qquad [17]$$

respectively.

SOME NUMERICAL RESULTS AND DISCUSSIONS

The method of calculating the flow field around a supercavitating cascade, as presented in the previous sections, is straightforward. The solution is in many respects similar to that given for isolated supercavitating hydrofoils. The calculations generally involve only numerical integrations and simple algebraic operations.

Computations were performed for blades with lower surface slope of the following types:

(I) $\dfrac{dy_f}{dx} = 0$ (flat-plate)

(II) $\dfrac{dy_f}{dx} = \dfrac{16}{9\pi} k \left(1 - 2\dfrac{x}{c}\right)$ (circular-arc camber)

(III) $\dfrac{dy_f}{dx} = \dfrac{4}{5\pi} k \left(1 + 4\sqrt{\dfrac{x}{c}} - 8\dfrac{x}{c}\right)$ (two-term camber)

(IV) $$\frac{dy_f}{dx} = \frac{2}{3\pi} k \left(1 - 6\sqrt{\frac{x}{c}} + 32\frac{x}{c} - 32\sqrt[3]{\frac{x}{c}}\right) \text{(three-term camber)}$$

(V) $$\frac{dy_f}{dx} = \frac{4}{5\pi} k \left(1 - 16\sqrt{\frac{x}{c}} + 120\frac{x}{c} - 368\sqrt[3]{\frac{x}{c}} + 512\frac{x^2}{c^2} - 256\sqrt[5]{\frac{x}{c}}\right) \quad \text{(five-term camber)}$$

The constant k in these equations is the camber coefficient which, in isolated cases, represents the so-called design lift coefficient (for details see Johnson, 1961). The effects of blade profiles on lift coefficient are substantial. Some of the typical calculations are shown in Figure 4. The results seem to indicate that higher term cambered blades give higher lift coefficients. It is to be noted however, that the forward portion of higher term cambered blade is very thin. The strength of blade structure may therefore become an important factor in the final design of the supercavitating cascade.

One of the primary interests in the cascade analysis is, of course, the behavior of lift coefficient with different cascade geometries. Figure 5 illustrates the lift coefficient for supercavitating cascades of flat plate and circular arc blades as functions of stagger angle and solidity. In general the cascade geometry has marked influence on cascade performance. The interference effect is larger for larger stagger angles and solidities.

The effect of cavity length on cascade performances is also of practical interest. Some of the theoretical calculations are depicted in Figures 6 and 7. It is found in general, that the lift coefficient and cavitation number decrease with increasing cavity length and approach rather rapidly to certain asymptotic values. Also shown in Figures 6a and 7a are experimental data of Wade and Acosta (1966) for supercavitating flat plate cascades. The theoretical predictions are in reasonably good agreement with experimental findings.

CONCLUDING REMARKS

In the present study a linearized theory of supercavitating flow past a straight cascade with arbitrary blade shapes is developed. From the analysis, it is possible to determine the lift and drag coefficients, cavitation number, cavity shape and exit flow conditions for any given specific cascade geometry, blade shape, cavity length and initial inflow conditions. The cavitating performance of the cascade is, in general, found to depend strongly on stagger angle, solidity, blade shape and cavity length.

It is needless to say that the present analysis is limited to cases in which the disturbance, caused by the presence of the blade, is small — an inherent restriction in the linear approximation. However, in practice the pump, turbine or propeller blades are quite thin and the linearized results obtained serve as a guide to the designer and aid in the interpretation of test results obtained for supercavitating pumps and turbines.

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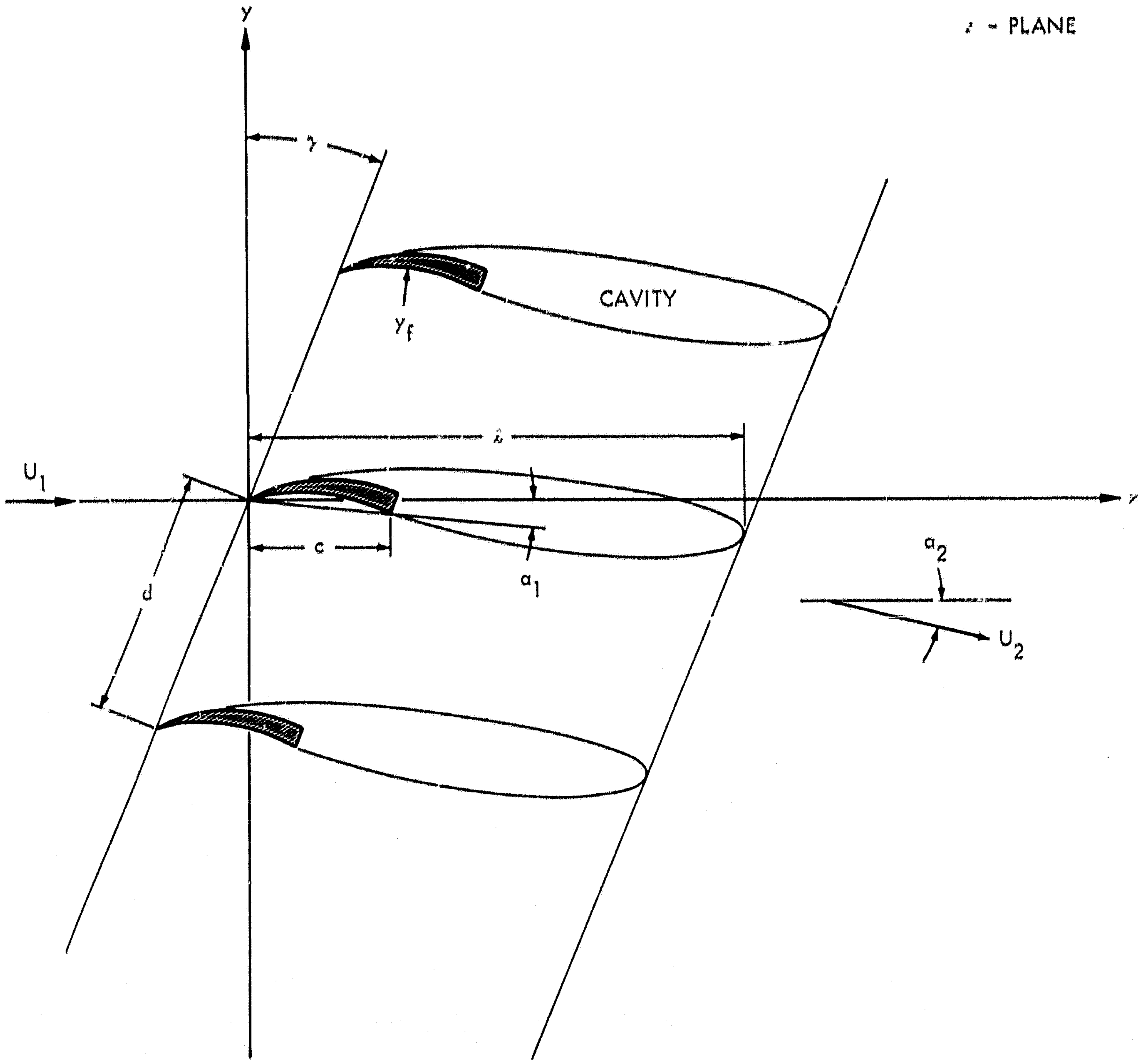

FIGURE 1 - DEFINITION SKETCH

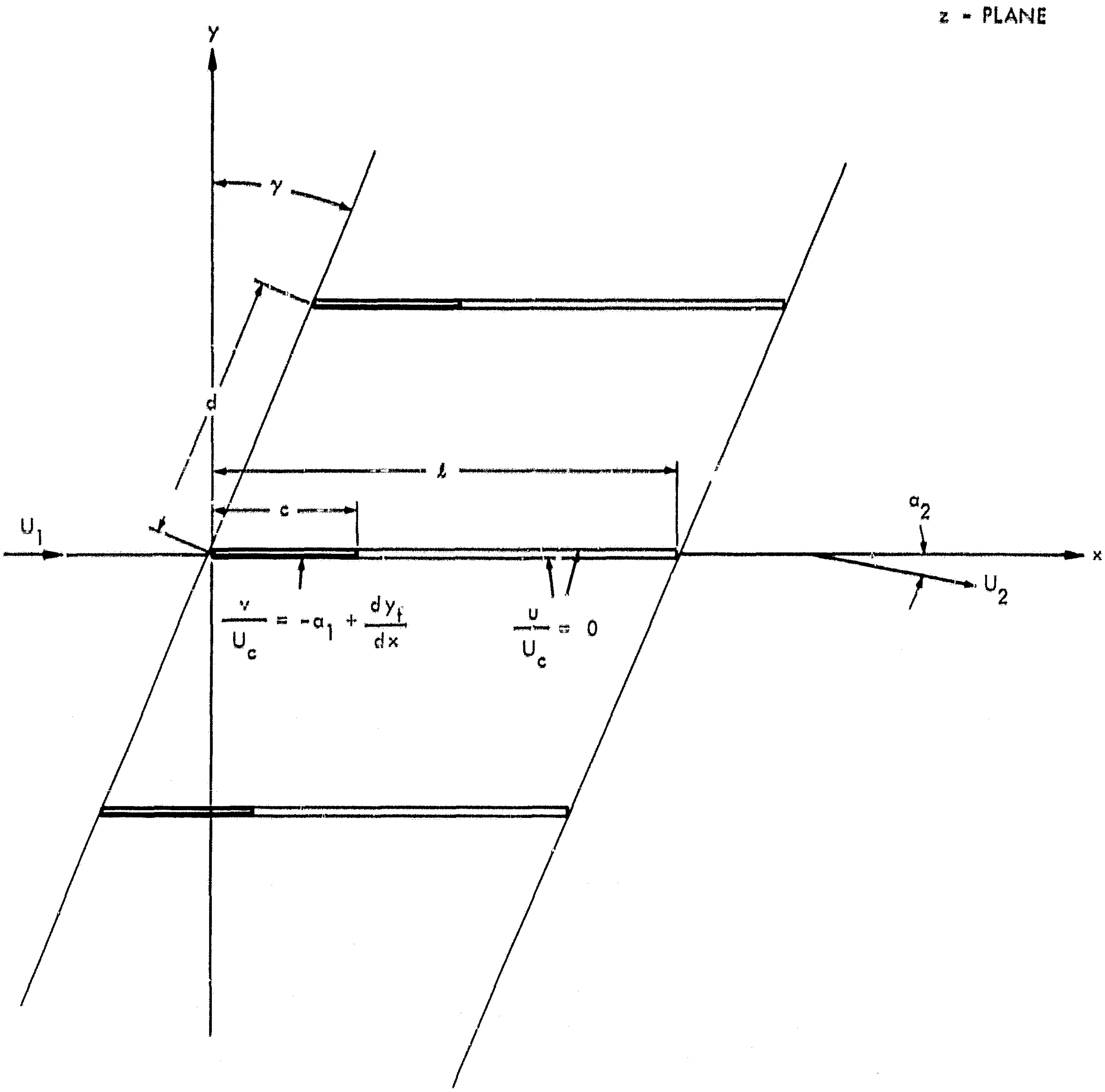

FIGURE 2 - LINEARIZED BOUNDARY VALUE PROBLEM IN THE PHYSICAL z-PLANE

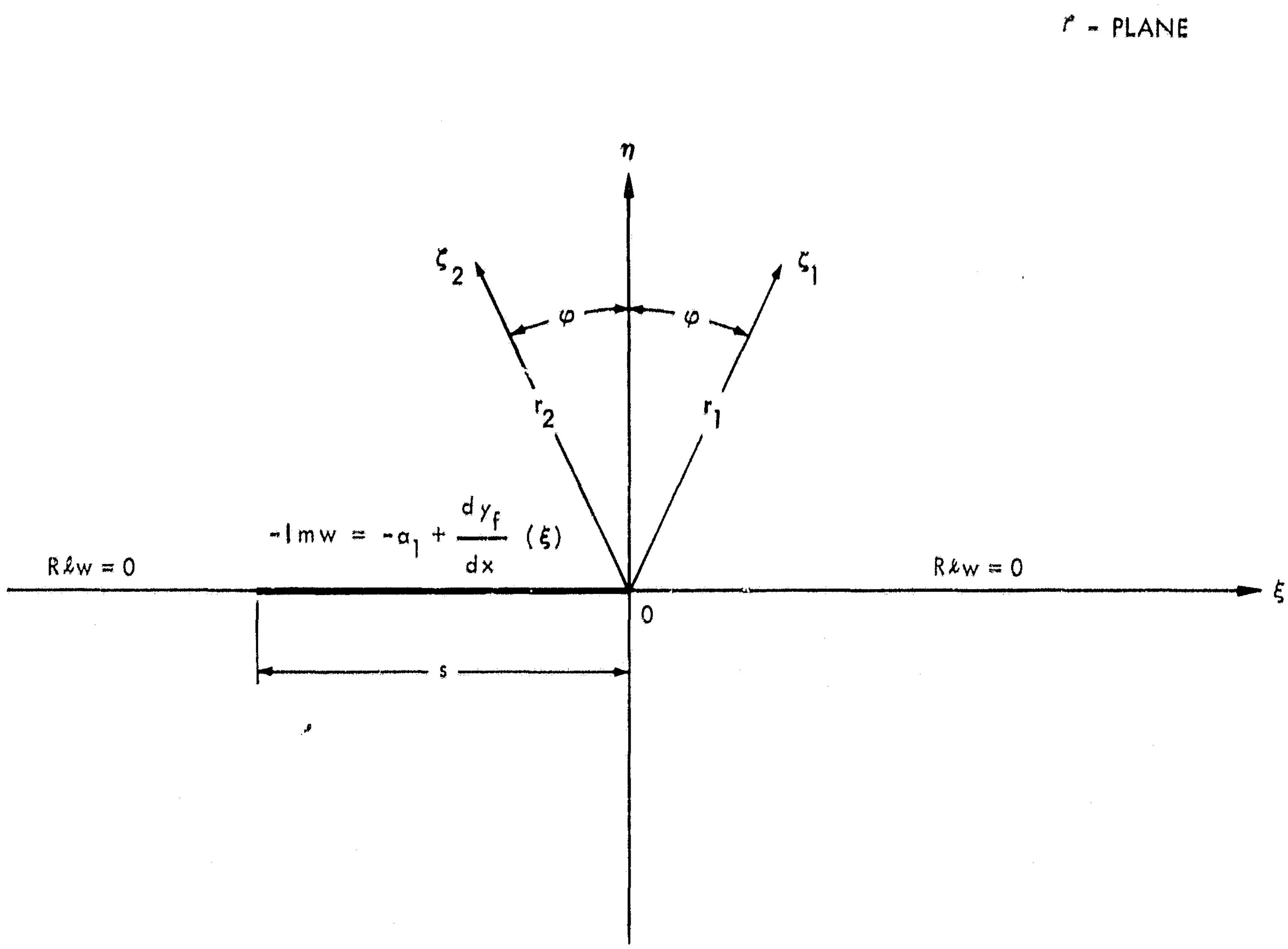

FIGURE 3 - LINEARIZED BOUNDARY VALUE PROBLEM IN THE TRANSFORMED ζ-PLANE

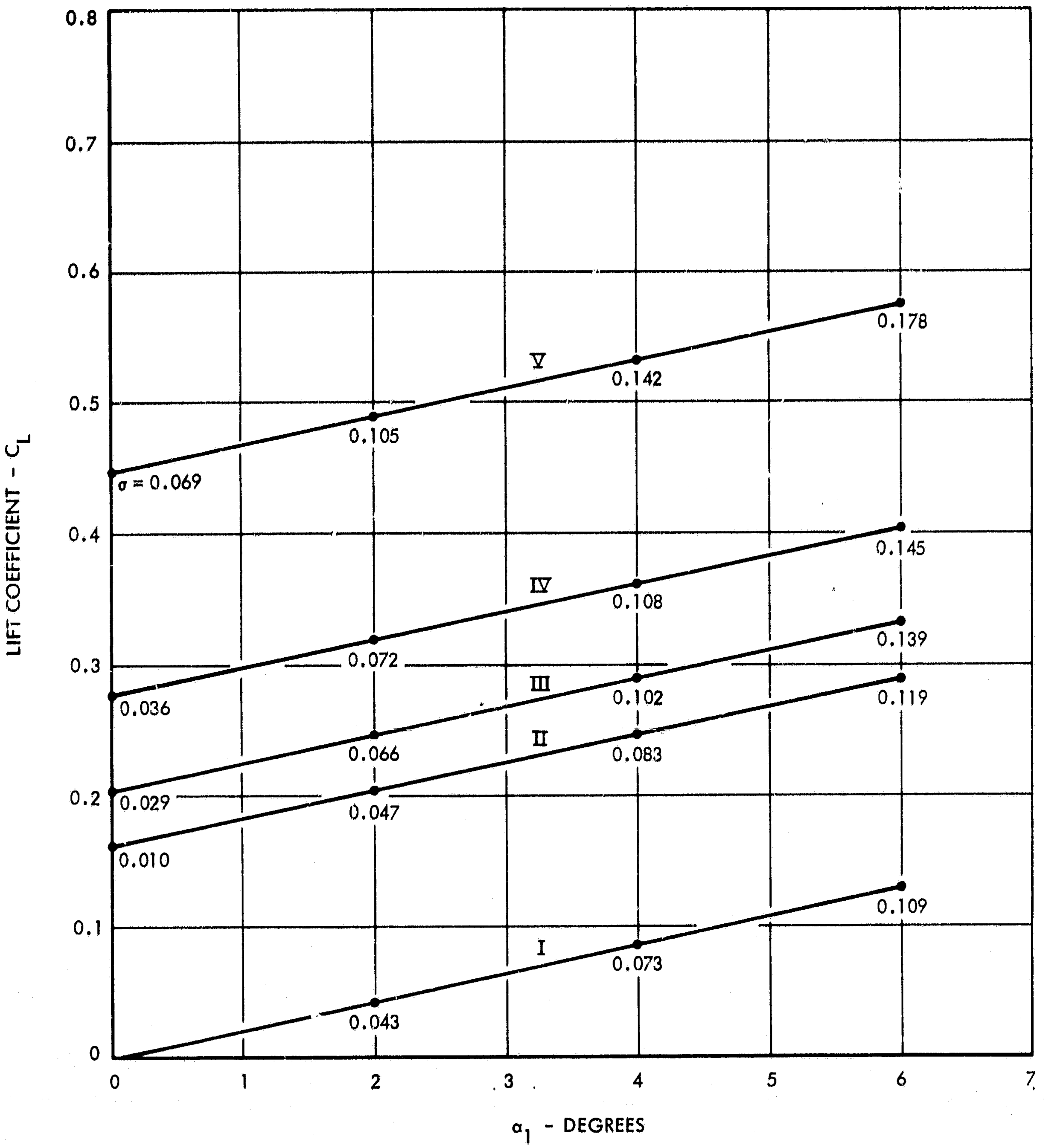

FIGURE 4 - LIFT COEFFICIENT FOR SUPERCAVITATING CASCADE WITH FLAT PLATE AND CAMBERED BLADES ($\gamma = 30^\circ$, $c/d = 0.51$, $\ell/c = 13.406$)

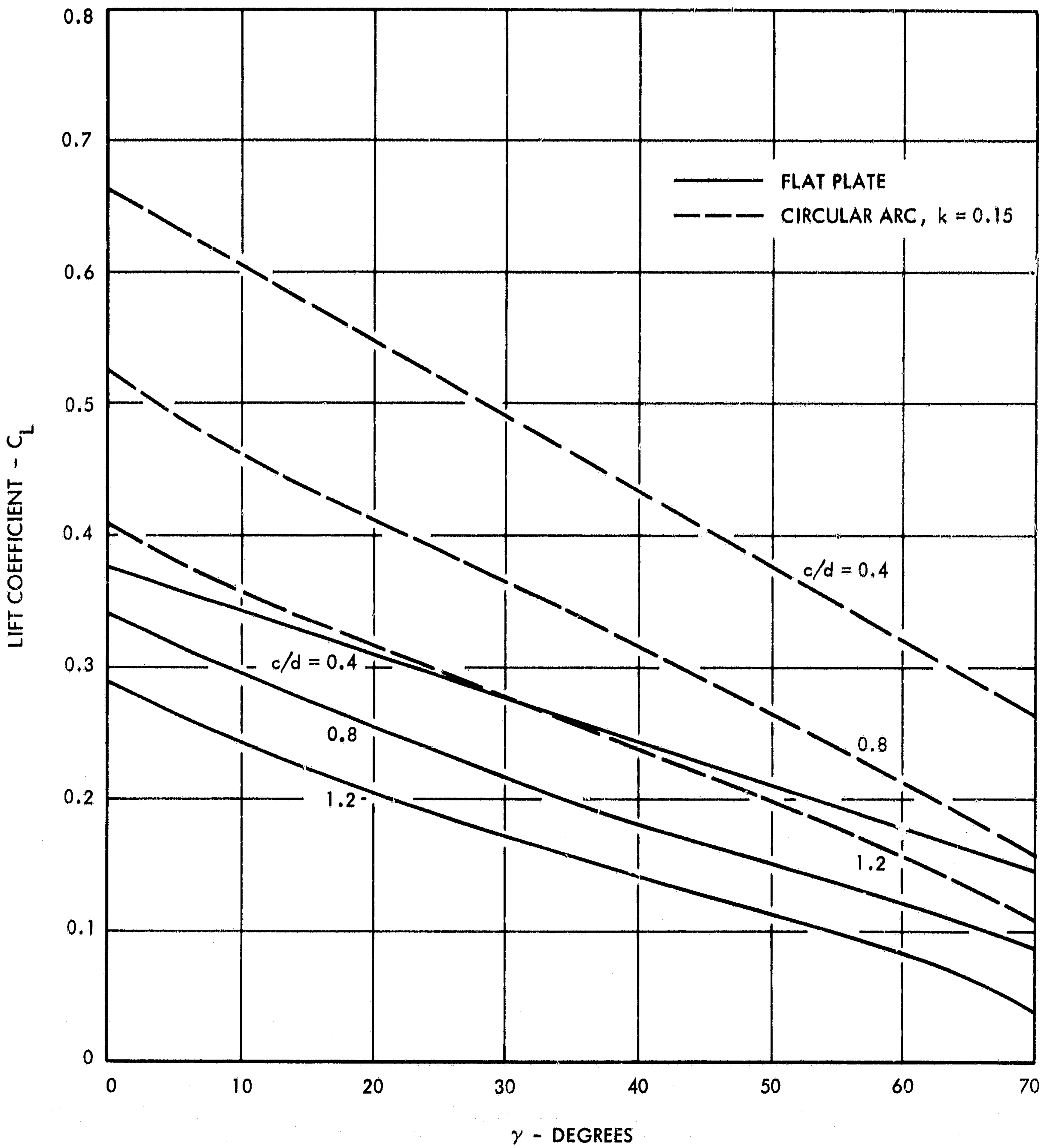

FIGURE 5 - LIFT COEFFICIENT FOR SUPERCAVITATING CASCADE WITH FLAT PLATE AND CIRCULAR ARC BLADES
($\alpha_1 = 10^\circ$, $b/a = 1.00001$)

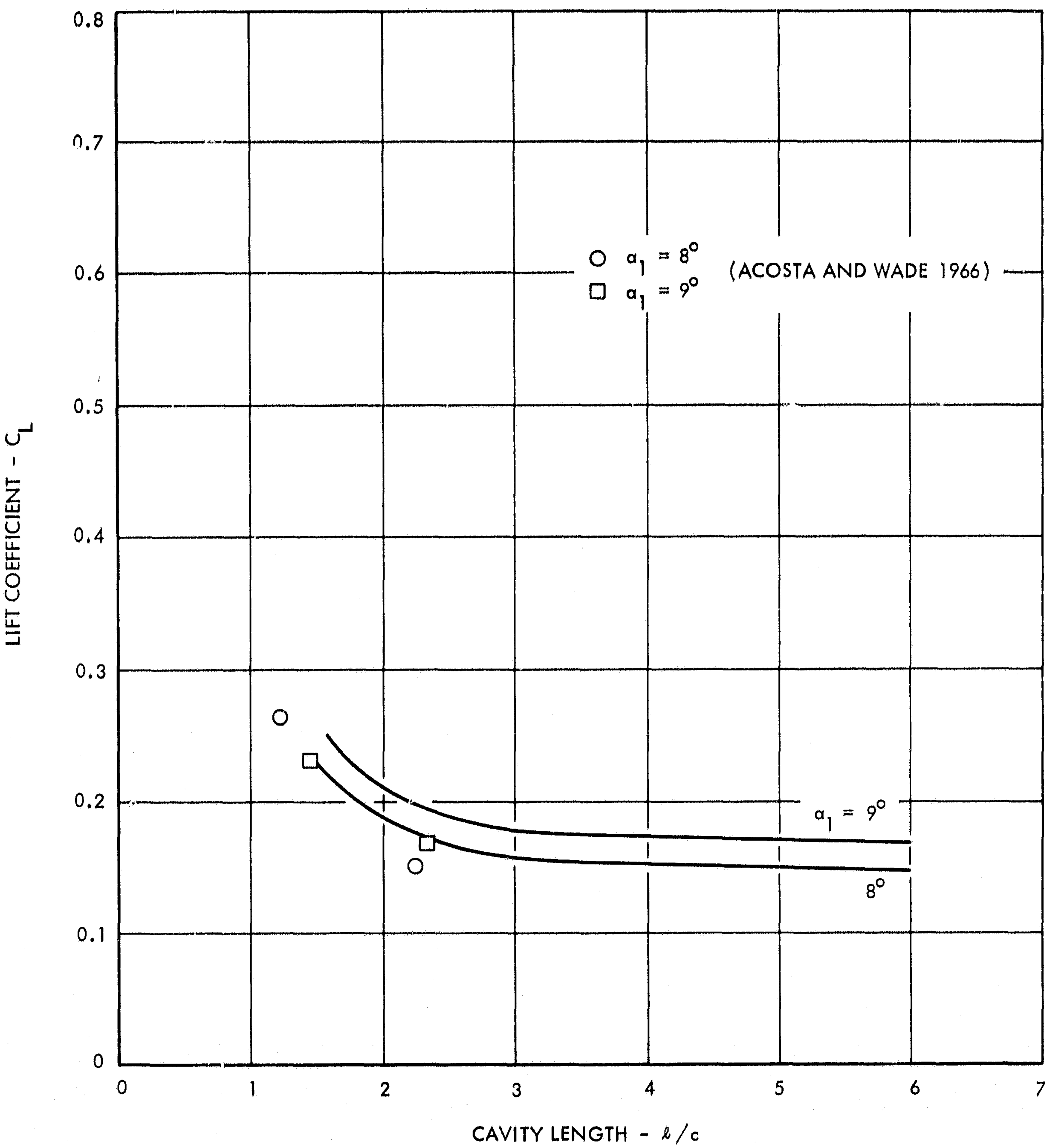

FIGURE 6a - LIFT COEFFICIENT FOR SUPERCAVITATING CASCADE WITH FLAT PLATE AS A FUNCTION OF CAVITY LENGTH ($\gamma = 45^\circ$, $c/d = 0.625$)

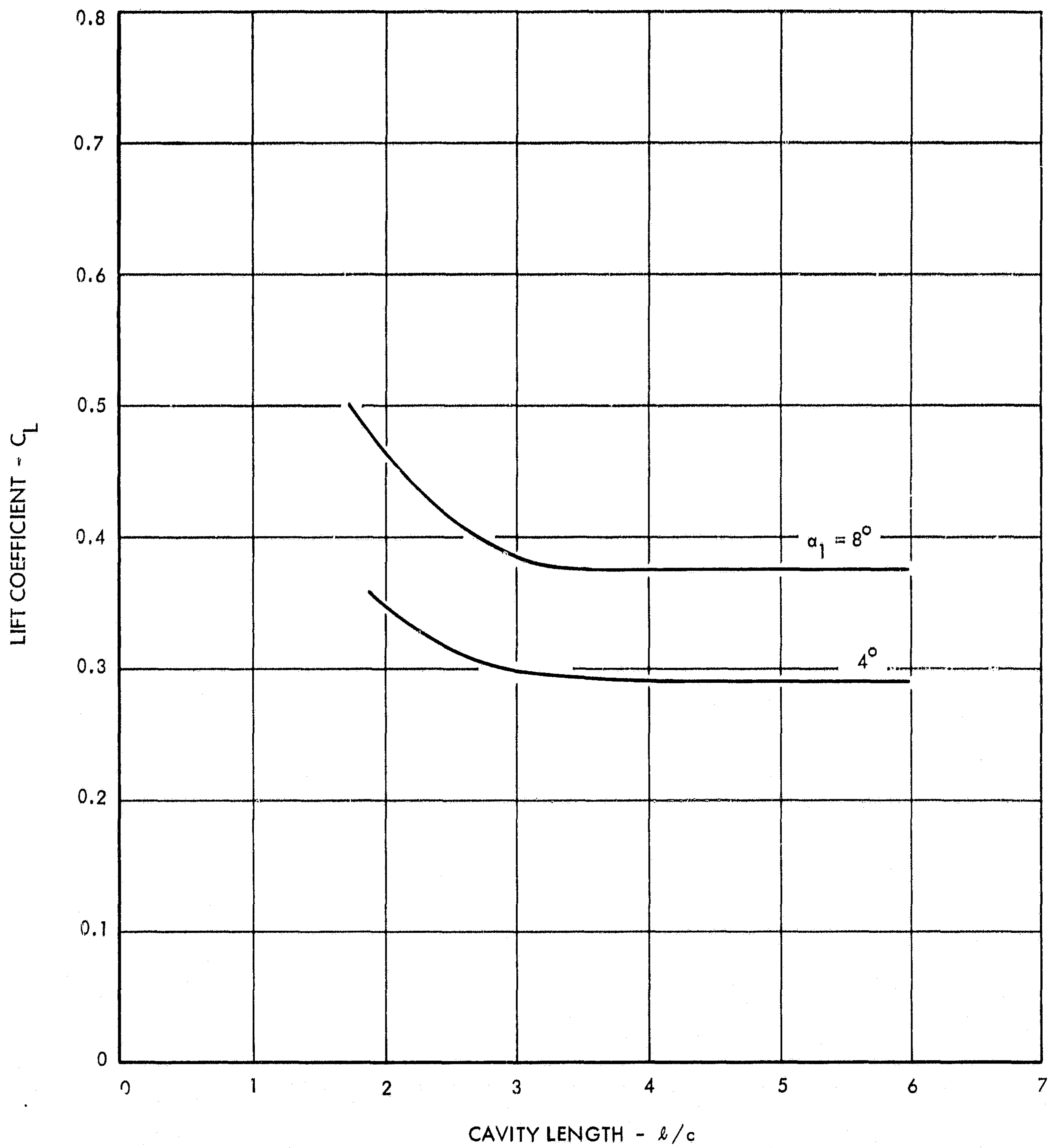

FIGURE 6b - LIFT COEFFICIENT FOR SUPERCAVITATING CASCADE WITH CIRCULAR ARC BLADES AS A FUNCTION OF CAVITY LENGTH
($k = 0.247$, $\gamma = 45°$, $c/d = 0.625$)

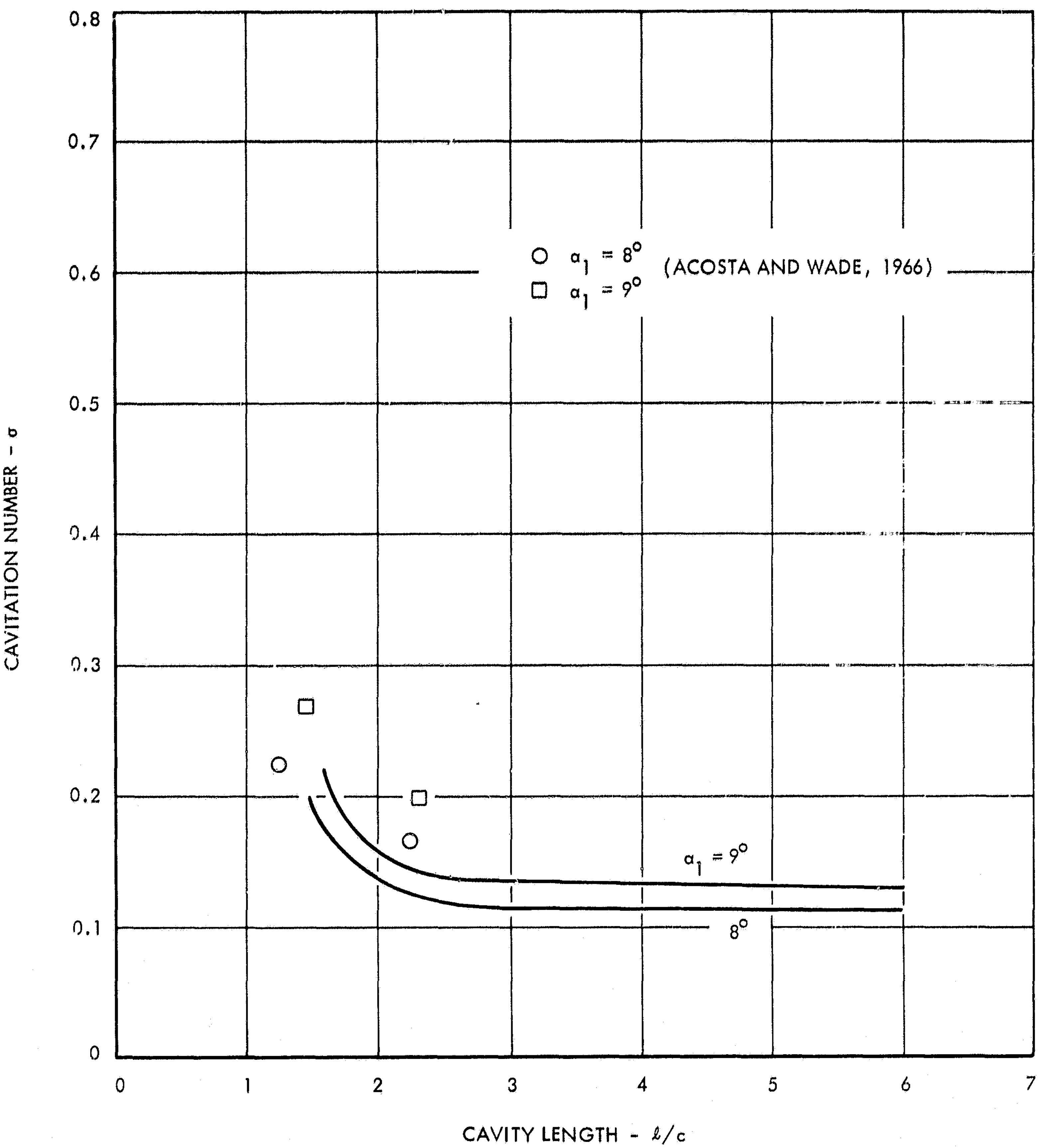

FIGURE 7a - CAVITATION NUMBER FOR SUPERCAVITATING CASCADE WITH FLAT PLATE AS A FUNCTION OF CAVITY LENGTH ($\gamma = 45^{\circ}$, $c/d = 0.625$)

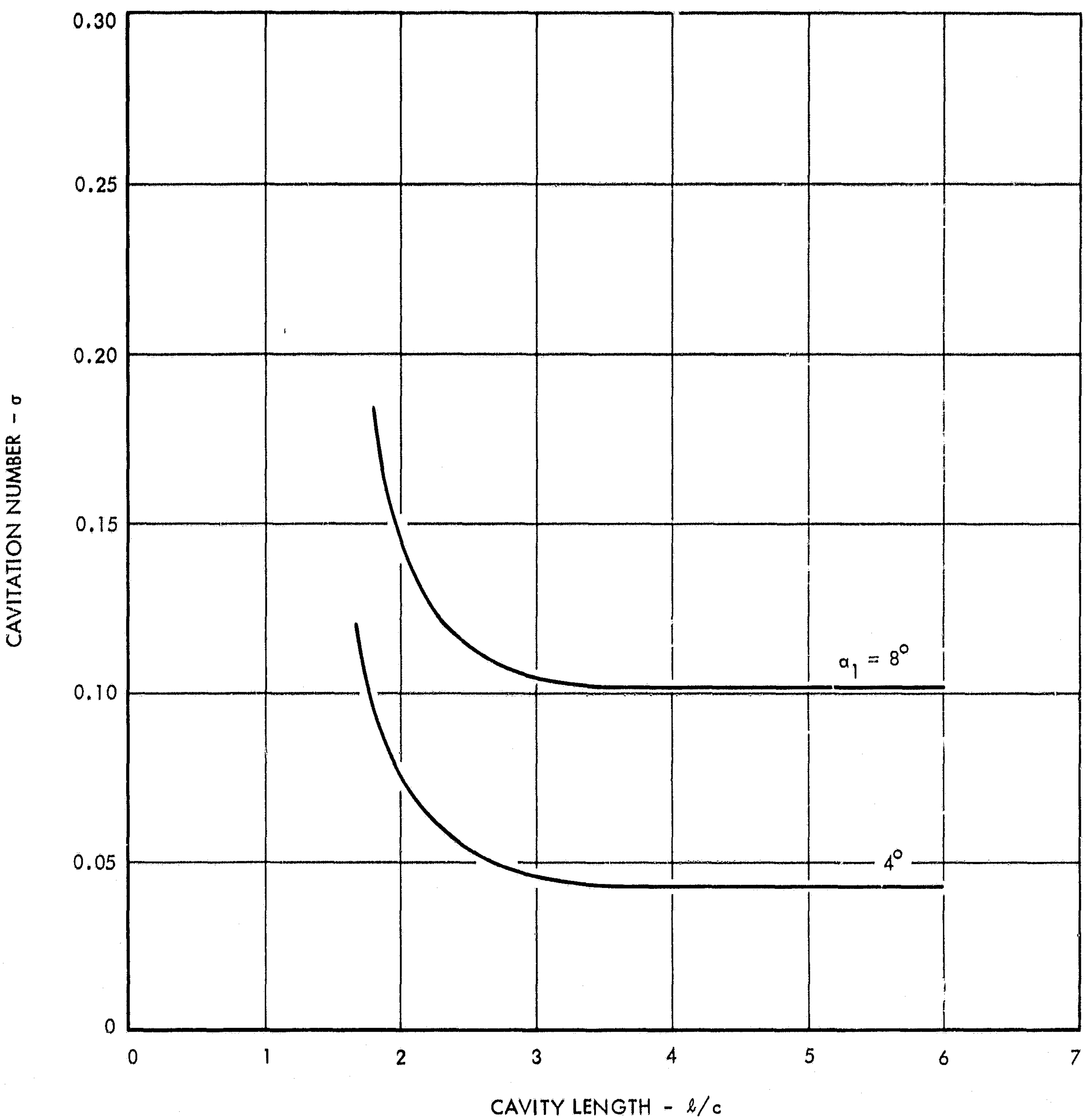

FIGURE 7b - CAVITATION NUMBER FOR SUPERCAVITATING CASCADE WITH CIRCULAR ARC BLADES AS A FUNCTION OF CAVITY LENGTH
($k = 0.247$, $\gamma = 45^\circ$, $c/d = 0.625$)